EDGE WAVES INDUCED BY A RADIALLY SPREADING LONG WAVE AND ITS DAMPING DUE TO THE IRREGULARITY OF COAST

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Abstract

First, the generation of edge waves by a cylindrically spreading long wave and, next, its damping due to the irregularity of coast are treated in two cases of the epicontinental bottom configuration such as is treated by Sezawa.

1. Introduction

The study of edge waves in water seems to be initiated by Stokes (1846) in case of water depth of a constant slope, and the extension to include the higher modes is made by Ursell (1952). Oceanographical application and development of this theory have been made to storm surges by Munk et al (1956) and Greenspan (1956). The effect of the constant Coriolis factor on the edge wave behavoir and their generation has been investigated by Reid (1958) and Kajiura (1958).

As stated above, however, their treatment of the problem is based on the water depth of linear slope. So their theory may not well be applied when onshore wave length is of comparable order to the width of the continental shelf, which will correspond, in the usual case, to the wave period of several ten minutes.

Now, the model of ocean bottom available to this latter case is found in a paper by Sezawa (1939), where the model is taken of the uniform water depth on and outside the shelf, and the name "epicontinental wave" is first originated.

Of the edge wave problems, the question may naturally arise whether edge waves exist or not in the tsunami case. The tsunami recorder near the coast shows multiply fluctuating wave trains altering the amplitude as well as the period. This may be partly ascribed to the dispersion nature of gravity waves or to the scattering of the wave energy by the strong irregularity of the ocean bottom. The essential point, however, will be that the existence of the continental shelf and bays are most responsible factors to the above described characteristics. In this connection, from the seismological Love wave analogy, the natural picture will be that the continental shelf might produce the edge waves of Love type in the tsunami case. Also, for an incoming solitary wave, the dispersive character of the edge waves will produce the waves, which propagate with the group velocity, changing periods and wave lengths as they propagate along the continent.

On the other hand, the strong possibility that the decay of wave energy in its propagation will be very large owing to the bottom friction and the non-linear breaking effect near the coast, might lead to the unimportance of the edge waves. Also, the irregularity of the coast must be added to the factors of decay process when it is very large, though it may necessarily associated with the irregularity of the ocean bottom.

The present paper is only a crude initial step to this problem and deals with the "epicontinental" case in both the generation and damping problem. The former half of the paper describes the generation of edge waves in the fundamental case when a periodic long wave spreads radially outwards,* and the latter half the damping of edge waves due to the irregularity of coast** after the similar manner as that of Cox (1956).***

In the present paper, two cases are considered concerning the bottom configuration, and adoptation of the first case is exclusively due to the mathematical simplicity in spite of its rather unrealistic features.

Lastly, it must be remarked that in the present paper the complete neglection of the breaking effect near the coast is made and instead, the condition of the total reflection at the coast is imposed of the onshore waves. Certainly, the results obtained here may be more or less altered when this effect is taken into account.

2. Generation

In this section, the problem will be treated when a primary incident long wave spreads radially outwards from the disturbance origin in cases where the bottom has a simplest configuration and then a more general form.

2.-1 Formulation and solution of the problem

In this article, the problem is dealt with in the simplest case when the shelf and the ocean offing of it are uniform as shown in Fig. 1. The coordinate axes are also shown in Fig. 1.

Equations for long waves are given by

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* Our first problem is quite similar to the generation of Love waves (Sezawa, 1935) and, of course, the results will be easily obtainable by changing the boundary conditions and by making an appropriate correspondence of the physical quantities. However, the analysis derived by the writer independently of Sezawa's will be described to keep a consistency througout the paper.

** This damping problem was kindly suggested by Dr. Cox.

*** One of Dr. Cox's works is on the conversion of a surface wave into internal ones due to the irregularity of the ocean bottom with special application to oceanic tides, use being made of the modern concepts of the wave power and the variance spectrum of the ocean bottom. These concepts will be utilized also in the present paper.

$$\frac{\partial u}{\partial t}=-g\frac{\partial \zeta}{\partial x},\qquad -\frac{\partial \zeta}{\partial t}=h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \tag{2-1}$$
$$\frac{\partial v}{\partial y}=-g\frac{\partial \zeta}{\partial y},$$

where u denotes the velocity in the x-direction, v in the y-direction, ζ the surface elevation, and h the undisturbed water depth, suffix 1 or 2 referring to the quantity in the region 1 or 2 respectively.

It is assumed that the center of a disturbance is located at (o, d) in the region 2, and the incident wave generated is expressed by

$$\zeta_{\text{incident}}=\frac{i}{\pi}A\int_{-\infty}^{+\infty}F(l)e^{+\alpha_l^{(2)}(y-d)-ilx+i\sigma t}\,dl,\quad y\leq d \tag{2-2}$$

Now the boundary conditions to be imposed are that the normal velocity at the boundary $y=0$ is zero, the volume transport and the surface elevation are both continuous at the edge of the shelf $y=L$, and the wave energy is finite at the infinity of y. Then, the solution for ζ is given by

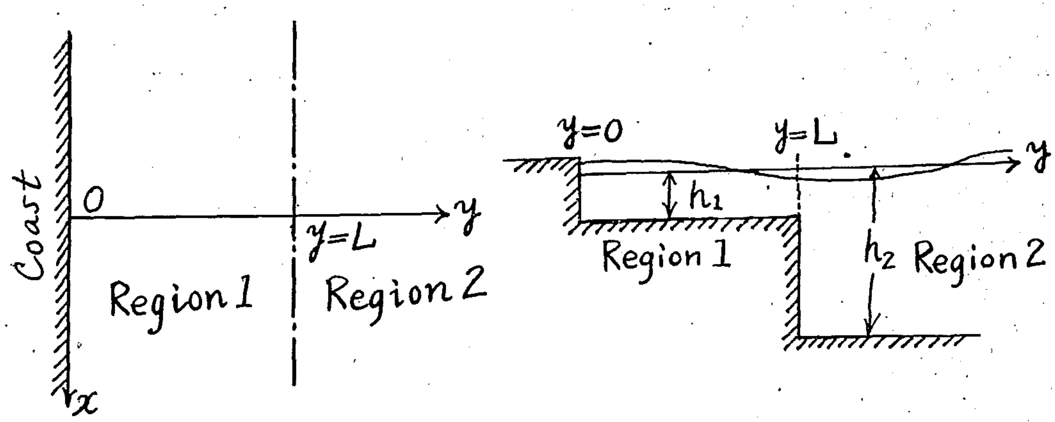

Fig. 1 Model of a continental shelf.

$$\left.\begin{aligned}\zeta_1&=\int_{-\infty}^{+\infty}(a_l e^{\alpha_l^{(1)}y}+b_l e^{-\alpha_l^{(1)}y})e^{i(\sigma t-lx)}\,dl,\\ \zeta_2&=\zeta_{\text{incident}}+\int_{-\infty}^{+\infty}c_l e^{-\alpha_l^{(2)}y+i(\sigma t-lx)}\,dl,\end{aligned}\right\} \tag{2-3}$$

where $\alpha_l^{(k)}=\sqrt{(l^2-q_k^2)}$, $q_k=\sigma/c_k$, $(k=1,2)$,

$$\left.\begin{aligned}a_l=b_l&=\frac{i}{\pi}A\alpha_l^{(2)}\cdot\frac{h_2}{h_1}\cdot F(l)\cdot e^{\alpha_l^{(2)}(L-d)}/G(l)\\ e^{-\alpha_l^{(2)}L}c_l&=-\frac{i}{\pi}AF(l)e^{\alpha_l^{(2)}(L-d)}+2\,a_l\cosh\alpha_l^{(1)}L\end{aligned}\right\} \tag{2-4}$$

and
$$G(l)=\alpha_l^{(1)}\sinh\alpha_l^{(1)}L+\frac{h_2}{h_1}\alpha_l^{(2)}\cosh\alpha_l^{(1)}L \tag{2-5}$$

Let us denote the real roots of $G(l)=0$ as $\pm l_1,\ \pm l_2,\cdots\cdots,\ \pm l_m$ which lie between

$|q_1|$ and $|q_2|$ in the absolute value.* Then edge wave solution of the n-th mode for the region $x>0$ is given by integral around the pole l_n as follows:

$$\left.\begin{aligned}\zeta_1&=4A\frac{h_2}{h_1}\alpha_{ln}^{(2)}F(l_n)\frac{e^{\alpha_{ln}^{(2)}(L-d)}}{G'(l_n)}\cosh\alpha_{ln}^{(1)}y\cdot e^{i(\sigma t-l_nx)}\\ \zeta_2&=4A\frac{h_2}{h_1}\cdot\alpha_{ln}^{(2)}\cdot F(l_n)\cdot\frac{e^{\alpha_{ln}^{(2)}(L-d)}}{G'(l_n)}\cdot\cosh\alpha_{ln}^{(1)}L\cdot e^{-\alpha_{ln}^{(2)}(y-L)+i(\sigma t-l_nx)}\end{aligned}\right\}\quad(2\text{-}6)$$

for sufficiently large x, where the dash on $G(l)$ denotes the differentiation with respect to l. (For the detail of integration method, see section 3.1)

Now, the power of the wave of the n-th mode averaged in time and space for positive x-direction is rendered by

$$\langle\overline{(pV)_x}\rangle=\frac{\rho g^2 l_n}{2\sigma}|\zeta|^2 \quad (2\text{-}7)$$

Then, the total power P^+_n of the above edge wave of the n-th mode given by

$$P_n^+=\int_0^\infty\langle\overline{(pV)_x}\rangle\, hdy=\frac{4\rho g^2 l_n A^2}{\sigma}\cdot\left(\frac{h_2}{h_1}\right)^2\cdot\frac{|\alpha_{ln}^{(2)}|^2\cdot|F(l_n)|^2}{|G'(l_n)|^2}\cdot e^{2\alpha_{ln}^{(2)}(L-d)}$$
$$\times\left[h_1\left(L+\frac{\sinh 2\alpha_{ln}^{(1)}L}{2\alpha_{ln}^{(1)}}\right)+h_2\cdot\frac{\cosh^2\alpha_{ln}^{(1)}L}{\alpha_{ln}^{(2)}}\right]\quad(2\text{-}8)$$

The edge wave solution for ζ and the total power P_n^- in the negative x-region are also obtained by replacing l_n with $-l_n$ in eqs (2-6) and (2-8) respectively.

The form of $F(l)$ and the power of the incident waves will be calculated in section 2. 3.

Next, the solution will be given when the disturbane center is located inside the region 1. We assume that the incident wave is given by

$$\left.\begin{aligned}\zeta_{\text{b. incid.}}&=\frac{i}{\pi}A\int_{-\infty}^{+\infty}F_b(l)e^{\alpha_l^{(1)}(y-d)-ilx+i\sigma t}\,dl,\quad y\leqq d,\\ \zeta_{\text{f. incid.}}&=\frac{i}{\pi}A\int_{-\infty}^{+\infty}F_f(l)e^{-\alpha_l^{(1)}(y-d)-ilx+i\sigma t}\,dl,\quad y\geqq d.\end{aligned}\right.\quad(2\text{-}9)$$

Then from the same boundary conditions as mentioned above, we have the solutions for the surface elevation ζ in the following form:

$$\zeta_1=2A\alpha_{ln}^{(1)}\cdot\frac{[F_f(l_n)e^{\alpha_{ln}^{(1)}d}+F_b(l_n)e^{-\alpha_{ln}^{(1)}d}]}{G'(l_n)}\cdot\left[\cosh\alpha_{ln}^{(1)}(y-L)\right.$$
$$\left.-\frac{h_2\alpha_{ln}^{(2)}}{h_1\alpha_{ln}^{(1)}}\sinh\alpha_{ln}^{(1)}(y-L)\right]e^{i(\sigma-l_nx)}$$

$$\zeta_2=2A\alpha_{ln}^{(1)}\cdot\frac{[F_f(l_n)e^{\alpha_{ln}^{(1)}d}+F_b(l_n)e^{-\alpha_{ln}^{(1)}d}]}{G'(l_n)}\cdot e^{-\alpha_{ln}^{(2)}(y-L)}\cdot e^{i(\sigma t-l_nx)}\quad(2\text{-}10)$$

Also the power formula for P_n^+ is given as follows:

* This means physically that the propagation velocity of edge waves is between the wave velocity inside the shelf and that in the open sea.

$$P_n^+ = \frac{\rho g^2 A^2 l_n}{\sigma} \cdot (\alpha_{ln}^{(1)})^2 \cdot \frac{|F_f(l_n) e^{\alpha_{ln}^{(1)} d} + F_b e^{-\alpha_{ln}^{(1)} d}|^2}{[G'(l_n)]^2}$$

$$\times \frac{1}{\cosh^2 \alpha_{ln}^{(1)} L} \left[h_1 \left(L + \frac{\sinh 2\alpha_{ln}^{(1)} L}{2\alpha_{ln}^{(1)}} \right) + h_2 \cdot \frac{\cosh^2 \alpha_{ln}^{(1)} L}{\alpha_{ln}^{(2)}} \right]. \qquad (2\text{-}11)$$

2.-2 Formulation and solution of the problem (general case)

Here, the problem will be treated in case when the bottom has a more general form. We divide the ocean bed into two regions — the continental shelf (region 1) and the region off the shelf (region 2) — as illustrated in Fig. 2. The water depth of the shelf has an arbitrary form without zero depth, and the sea off the shelf is of constant depth, and is continuous at the separation point $y=L$. The coordinate axes are also shown in Fig. 2.

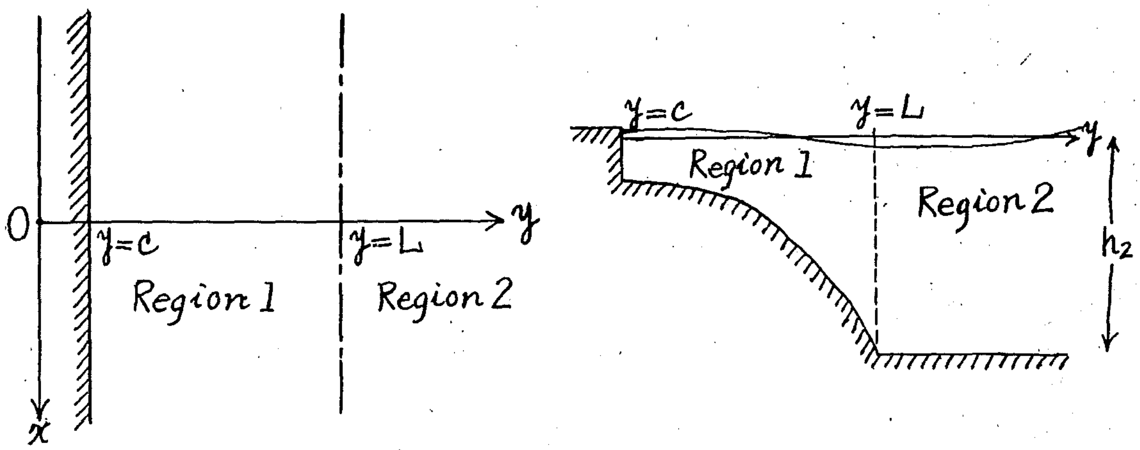

Fig. 2 Model of a continental shelf.

Equations for long waves are same as in Sec. 2.-1. except the continuity condition which is given in this case by

$$-\frac{\partial}{\partial t}\zeta = \frac{\partial}{\partial x}(uh) + \frac{\partial}{\partial y}(vh)$$

Here it is assumed that the center of the disturbance is located at (o, d) in region 2.

Now, the solution in the region 1 is given by

$$\zeta_1 = \int_{-8}^{+\infty} e^{i(\sigma t - lx)} (a_l Y_l^{(1)}(y) + b_l Y_l^{(2)}(y) dl, \qquad (2\text{-}12)$$

where two mutually independent solutions $Y_l^{(k)}(y)$ $(k=1, 2)$ satisfy the equation:

$$\frac{d^2\zeta}{dy^2} + \frac{1}{h} \cdot \frac{dh}{dy} \cdot \frac{d\zeta}{dy} + \left(\frac{\sigma^2}{gh} - l^2 \right) \zeta = 0 \qquad (2\text{-}13)$$

In the region 2, except the incident wave it is sufficient for us only to consider the solution of the form

$$\zeta_2(\text{sec}) = \int_{-\infty}^{+\infty} c_l e^{-\alpha_l y + i(\sigma t - lx)} \tag{2-14}$$

where $\alpha_l = \sqrt{(l^2 - q_2^2)}$, $q_2 = \sigma / c_2$, $c_2^2 = gh_2$ and $R_e \alpha_l \geqq 0$.

Here, the incident wave is supposed, as in section 2.-1, to have the form (2-2). The boundary conditions is the same as in the preceeding section. Then, the coefficients of eqs (2-12) and (2-14) are given by

$$\begin{aligned} a_l &= -\frac{2iA}{\pi} \cdot \alpha_l F(l) \frac{Y_l^{(2)\prime}(c)}{G(l)} \cdot e^{\alpha_l(L-d)} , \\ b_l &= +\frac{2iA}{\pi} \cdot \alpha_l F(l) \frac{Y_l^{(1)\prime}(c)}{G(l)} \cdot e^{\alpha_l(L-d)} , \\ c_l &= +\frac{2iA}{\pi} \cdot \alpha_l F(l) \frac{e^{\alpha_l(2L-d)}}{G(l)} \cdot [Y_l^{(1)\prime}(c) Y_l^{(2)}(L) - Y_l^{(1)}(L) Y_l^{(2)\prime}(c)] \\ &\quad - \frac{iA}{\pi} \cdot F(l) e^{\alpha_l(2L-d)} \end{aligned} \tag{2-15}$$

where

$$\begin{aligned} G(l) &= Y_l^{(1)\prime}(c) Y_l^{(2)\prime}(L) - Y_l^{(1)\prime}(L) Y_l^{(2)\prime}(c) \\ &\quad - \alpha_l [Y_l^{(1)}(L) Y_l^{(2)\prime}(c) - Y_l^{(1)\prime}(c) Y_l^{(2)}(L)] . \end{aligned} \tag{2-16}$$

Suppose that the real roots of $G(l)=0$ are $\pm l_1$, $\pm l_2$, $\cdots\cdots$($|l_n| \geqq q_2$).* Here it must be remarked that only the real roots of eq (2-16) are considered, since otherwise evanescent or growing waves will be obtained and the latter cannot be accepted from the physical point of view.

Then, we have the edge wave solution of the n-th mode by intergation around the pole l_n in the following form

$$\begin{aligned} \zeta_1 &= -4A \cdot \frac{\alpha_{ln} F(l_n)}{G'(l_n)} \cdot e^{\alpha_{ln}(L-d)} \cdot [Y_{ln}^{(2)\prime}(c) Y_{ln}^{(1)}(y) \\ &\quad - Y_{ln}^{(1)\prime}(c) Y_{ln}^{(2)}(y)] \cdot e^{i(\sigma t - l_n x)} , \\ \zeta_2 &= -4A \cdot \frac{\alpha_{ln} F(l_n)}{G'(l_n)} \cdot e^{\alpha_{ln}(L-d)} \cdot [Y_{ln}^{(1)}(L) Y_{ln}^{(2)\prime}(c) \\ &\quad - Y_{ln}^{(1)\prime}(c) Y_{ln}^{(2)}(L)] \cdot e^{i(\sigma t - l_n x) - \alpha_{ln}(y-L)} , \end{aligned} \tag{2-17}$$

for positive large x. The solution for the negative x region will be obtained by replacing l_n with $-l_n$.

Now, the edge wave of the n-th mode is given by**

$$\zeta_n = \varphi_n \cdot e^{i(\sigma t - l_n x)} \tag{2-18}$$

where

* To our present problem, the symmetry axis will be the y- axis. Hence we conclude that the real roots of $G(l)=0$ are $\pm l_1$, $\pm l_2$, $\cdots$. Moreover, from the above condition, we may say that $G(l)$ is a function of l^2 and so $G'(-l) = -G'(l)$.

** This expression is derived easily by imposing the condition that the normal velocity is zero at the coast, the volume transport as well as the surface elevation is continuous at y=L, and the evanescence of these quantities at the infinity of y.

$$\varphi_n(y)=\begin{cases} c_{ln}\cdot\dfrac{\alpha_{ln}}{H_{ln}}\cdot e^{-\alpha_{ln}}\cdot[-Y_{ln}^{(2)\prime}(c)Y_{ln}^{(1)}(y)+Y_{ln}^{(1)\prime}(c)Y_{ln}^{(2)}(y)],\ 0\leqq y\leqq L,\\ c_{ln}\cdot e^{-\alpha_{ln}y} \qquad L\leqq y \end{cases}$$

$$H_{ln}=Y_{ln}^{(1)\prime}(L)Y_{ln}^{(2)\prime}(c)-Y_{ln}^{(1)\prime}(c)Y_{ln}^{(2)\prime}(L), \tag{2-19}$$

and l_n is the real root of eq (2-16).

Then the solution (2-17) is unified to a single expression

$$\zeta=-4A\cdot\alpha_{ln}\cdot F(l_n)\cdot\frac{e^{\alpha_{ln}(L-d)}}{G'(l_n)}\cdot\frac{[Y_{ln}^{(2)\prime}(c)Y_{ln}^{(1)}(L)-Y_{ln}^{(1)\prime}(c)Y_{ln}^{(2)}(L)]}{\varphi_n(L)}\cdot\varphi_n(y)\cdot e^{i(\sigma t-l_nx)}, \tag{2-20}$$

for positive x.

If we set as

$$\int_0^{+\infty}h\,|\varphi_n|^2dy=1, \tag{2-21}$$

then $\quad |\varphi_n(L)|^2=\dfrac{2\,l_n}{h(L)}\cdot\dfrac{H_{ln}/\alpha_{ln}}{G'(l_n)}$.

Hence, we have P_n^+ as follows

$$P_n^+=\frac{4\,\rho g^2A^2}{\sigma}\cdot(\alpha_{ln})^3\cdot|F(l_n)|^2\cdot h(L)\cdot e^{-2\alpha_{ln}(d-L)}\cdot\frac{|Y_{ln}^{(2)\prime}(c)Y_{ln}^{(1)}(L)-Y_{ln}^{(2)\prime}(c)Y_{ln}^{(2)}(L)|^2}{G'(l_n)\cdot H_{ln}}, \tag{2-22}$$

Also, we have the power P_n^- in the negative x-direction, replacing l_n by $-l_n$ in eq (2-22), which is equal each other in the absolute value.

2.-3 The form of $F(l)$ and the power of the primary incident wave

In this section, the function $F(l)$ will be determined for further calculation based upon the long wave theory in case when a periodic long wave spreads radially outwards with circular symmetry. Since the incident wave is of a circular symmetry, then we have the equation for a long wave in the following form, eliminating the time factor $e^{i\sigma t}$.

$$\frac{\partial^2\zeta}{\partial r^2}+\frac{1}{r}\frac{\partial\zeta}{\partial r}+q_2^2\zeta=0 \tag{2-23}$$

where r is the distance measured from (o, d). Hence the solution of the incident wave is rendered by the 0-th order Hankel function of the 2nd kind

$$\zeta_{\text{inc.}}=AH_0^{(2)}(q_2r) \tag{2-24}$$

For further calculation, we need to transform $H_0^{(2)}(q_2r)$ into the following form

$$H_0^{(2)}(q_2r)=\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{1}{\alpha_l}\cdot e^{-\alpha_ly'-ilx'}\,dl \quad \text{for} \quad y'>0, \tag{2-25}$$

where $x'=x$ and $y'=y-d$ (This relation is proved, for instance, by Nakano (1925)). Likewise, in the region $y<d$ we have

$$H_0^{(2)}(q_2r)=\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{1}{\alpha_l}\cdot e^{\alpha_l(y-d)-ilx}\,dl, \qquad (2\text{-}26)$$

and $F(l)$ in the preceeding sections is given by

$$F(l)=\frac{1}{\alpha_l}. \qquad (2\text{-}27)$$

Asymptotic expansion of $\zeta_{\text{inc.}}$ gives the result

$$\zeta_{\text{inc.}}=AH_0^{(2)}(q_2r)\simeq A\left(\frac{2}{\pi q_2r}\right)^{1/2} e^{-i\left(q_2r-\frac{1}{4}\pi\right)}. \qquad (2\text{-}28)$$

Hence the mean power of the wave in the r-direction is given by

$$\overline{pv_r}=\frac{\rho g^2q_2}{2\sigma}|\zeta|^2=\frac{\rho g^2A^2}{\pi\sigma}\cdot\frac{1}{r} \quad \text{for large } r,$$

and the total power $P_{\text{inc.}}$ of the wave is expressed by

$$P_{\text{inc.}}=\oint h_2\overline{pv_r}\,ds=\frac{2\rho g^2A^2h_2}{\sigma}. \qquad (2\text{-}29)$$

2.-4 Numerical examples and discussions

So far, we have reduced the edge wave solution induced by the primary disturbance, and the total powers for these waves are given. The power ratio of the edge waves generated to the primary wave will have the significant importance, since the power is interpreted as the energy flow. Also, the detailed kinematics on edge waves will give simple relations among the energy flow, total energy, wave energy etc., although these are omitted here. On the other hand, the amplitude ratios at the coast among the edge waves, directly arriving waves, etc. will have important significance, considering that the observations have been exclusively made at the coast.

Here, some numerical examples will be illustrated of sec. 2.-1 and 2.-2.

(a) Example of sec. 2.-1

As numerical constants, we take T (wave period) $=20$ min, $h_1=1000$ m, $h_2=4000$ m and $L=60$ km. The disturbance center is assumed to locate in region 2. In this case, only one mode with $l_1=4.7\times10^{-7}$ exists of the edge waves. Then we have the power ratio

$$P^{+}_{n=1}/P_{\text{inc.}}=6.38\times10^{-2}\cdot\exp[-2\,\alpha_{l1}^{(2)}(d-L)]$$

Since $P_{\text{inc.}}$ is obtained by integration on the complete circle around the disturbing center, the effective ratio of the power conversion will be four times the above, that is,

$$e.\,r.\,c.\approx25\times\exp[-2\,\alpha_{l1}^{(2)}(d-L)] \qquad \%$$

The damping coefficient is, in this case, given by $2\,\alpha_{l1}^{(2)}\approx7.6\times10^{-7}$ and the distance to decay as e^{-1} is $d-L\simeq10$ km.

(b) Example of sec. 2.-2

An example is given in case when $h=by^2$ and $\nu=0$ (see Sec. 3-2). As numerical constants, we take $T\approx 52$ min., $h_1\approx 160$ m, $h_2=4$ km, $c=10$ km, $L=$ 50 km and the disturbance center is assumed to locate in region 2. In this case the existing mode of edge waves is only one with $l_1=1.1\times 10^{-7}$ within the long wave range. This case gives the effective conversion ratio as follows

$$4\times\frac{P^{+}_{n=1}}{P_{\text{inc.}}}=9.7\times\exp[-2\,\alpha^{(2)}_{l_1}(d-L)] \qquad \%,$$

and the distance to decay as e^{-1} is about 100 km.

In both examples, it is remarked that the conversion ratio has an exponential damping factor with respect to the distance $d-L$, and it is largest when the disturbance center is at the edge of the shelf. This may well correspond to one of the results by Greenspan (1956). The conversion ratio will, however, be largely dependent to the bottom structure as well as the wave period.

3. Damping due to the Irregular Coast

Recently, Dr. Cox (1956) has made the investigation on the conversion damping of surface waves into the internal ones with the special application to the oceanic tides. This conversion is due to the irregular undulation of the ocean bottom.

Quite the similar things will happen with the edge waves. That is, the edge wave of a certain mode will be converted into the ones of the various modes when the wave travels along the coast having irregular undulation. In the present section, this problem will be treated based upon the perturbation procedure with respect to the amplitude of the irregular coast after the method by him.

This section is divided into two parts: first the special case, and next the general case as in sec. 2.

3.-1 Damping in the simplest case

3.-1.1 The formal solution of the problem

In this article, the problem is treated in the simplest case when an ocean bed is uniformly deep with a continental shelf of uniform depth, a coastal line being irregular in a finite extent with a basic straight line (Fig. 3). We take the coordinate axes as shown in Fig. 3.

Equations for a long wave are the same as eq (2-1).

Now, we set the deviation of the coastal boundary from the base line as

$$y=\varepsilon\lambda(x) \tag{3.1}$$

where ε denotes the perturbation parameter equal to unity. Then we expand all the relevant quantities in the power series of ε as follows:

$$\begin{aligned} \varPsi&=\varPsi_0+\varepsilon\varPsi_1+\varepsilon^2\varPsi_2+\cdots\cdots, \\ \zeta&=\zeta_0+\varepsilon\zeta_1+\varepsilon^2\zeta_2+\cdots\cdots. \end{aligned} \tag{3.2}$$

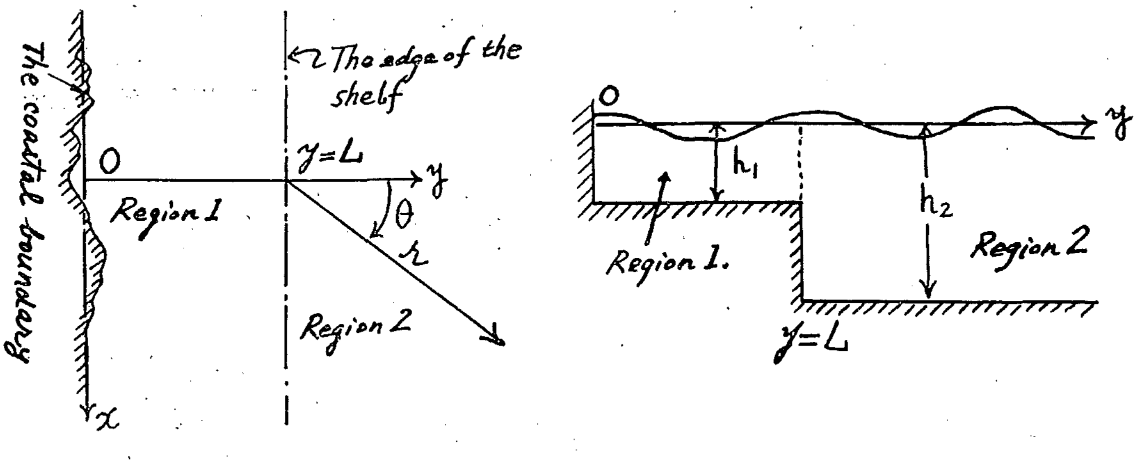

Fig. 3 Model of an epicontinent.

The consition to be satisfied at the coast is given by

$$\frac{D}{Dt}(\varepsilon\lambda(x)-y)\Big|_{\text{bdry}}=0 \text{ or } \left[\varepsilon u\frac{d\lambda}{dx}-v\right]_{\text{bdry}}=0. \tag{3-3}$$

Expanding the values in eq (3-3) at the coast in Taylor series, substituting them into eq (3-3) and then equating the same power of ε, we have the following conditions:

$$\varepsilon^0: \quad v_0(0)=0, \tag{3-4}$$

$$\varepsilon^1: \quad u_0(0)\frac{d\lambda}{dx}-v_1(0)-\lambda\frac{\partial v_0}{\partial y}\Big|_0=0, \tag{3-5}$$

$$\varepsilon^2: \quad \frac{d\lambda}{dx}\left\{u_1(0)+\lambda\frac{\partial u_0}{\partial y}\Big|_0\right\}-v_2(0)-\lambda\frac{\partial v_1}{\partial y}\Big|_0-\lambda^2\frac{\partial^2 v_0}{\partial y^2}\Big|_0=0,$$

and so on.

In the following, the quantities outside the shelf are denoted by subscript 2 or nothing and those inside by the subscript 1 or dash.

Now, we assume that the O-th order edge wave is given by

$$\zeta=B\cos s_1 L e^{i(\sigma t-l_0 x)-s_2(y-L)},$$
$$\zeta'=Be^{i(\sigma t-l_0 x)}\cos s_1 y, \tag{3-6}$$

where $\quad s_1{}^2=\frac{\sigma^2}{c_1{}^2}-l_0{}^2, \quad s_2{}^2=l_0{}^2-\frac{\sigma^2}{c_2{}^2}, \quad c_1{}^2=gh_1, \quad c_2{}^2=gh_2$

and $\quad \tan s_1 L=\frac{h_2 s_2}{h_1 s_1}.$

Terms of ε no lower than the 1st power in eq (3-2) represent the generated waves by the finite patch of the rough boundary in passing of the incident wave. The 1st order generated long waves can be calculated by the boundary condition (3-5), the conditions of continuity of surface elevation ζ and the volume transport vh at the edge of the continental shelf $y=L$, and the condition that the wave energy must be finite at $y=\infty$, resulting in the following formal expressions for the surface elevations ζ and ζ':

$$\zeta_1 = iB\int_{-\infty}^{+\infty} \frac{(l_0k+q_1^2)S(l_0+k)}{\left(\frac{h_2}{h_1}\right)\alpha_2\cos\alpha_1L+i\alpha_1\sin\alpha_1L}\cdot e^{ikx-i\alpha_2(y-L)}\,dk, \quad (3\text{-}7)$$

$$\zeta_1' = iB\int_{-\infty}^{+\infty} \frac{(l_0k+q_1^2)S(l_0+k)}{\left(\frac{h_2}{h_1}\right)\alpha_2\cos\alpha_1L+i\alpha_1\sin\alpha_1L}\cdot e^{ikx-i\alpha_2L}$$

$$\cdot\left[\cos\alpha_1(L-y)-i\frac{h_2\alpha_2}{h_1\alpha_1}\sin\alpha_1(L-y)\right]dk \quad (3.8)$$

where

$$\alpha_i=\sqrt{(q_i^2-k^2)},\quad q_i=\sigma/c_i\ (i=1,\ 2)\ \text{and}\ \lambda(x)=\int_{-\infty}^{+\infty}S(\mu)e^{ix\mu}d\mu.$$

Throughout all the calculation in this paper, we assume that the spectrum $S(\mu)$ of the boundary is continuous and also has no branch point.

3.-12 Integration of the expressions (3-7) and (3-8)

We need here to introduce a virtual friction in order to avoid the indetermination of the integrals (3-7) and (3-8). Also, to let the energy of waves be finite everywhere from the physical consideration and the integrands in (3-7) and (3-8) be one-valued functions of k, we choose as the plane of integration the top leaf of the Riemenn surfaces such that $Re\sqrt{(k^2-q_1^2)}\geqq 0$ and $Re\sqrt{(k^2-q_2^2)}\geqq 0$ after the method by Lapwood (1949). Taking this friction into consideration, we must lay the cuts as shown in Fig. 4. Poles of the integrands on the real axis in (3-7) lie in the interval $|q_1|\leqq|k|\leqq|q_2|$, these locations being denoted by $l_1<l_2<\cdots\cdots<l_m$.

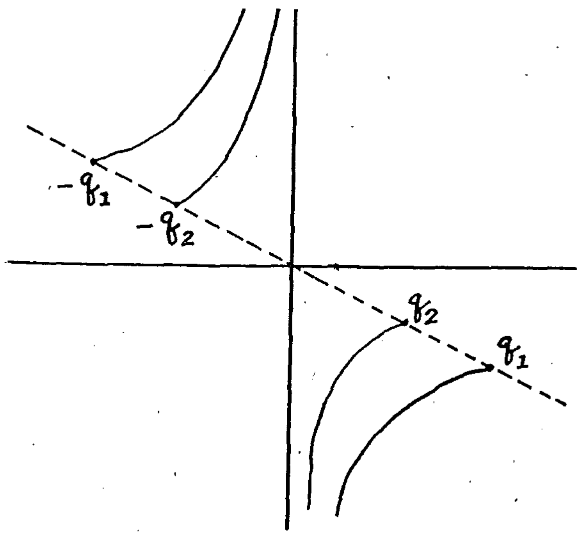

Fig. 4 Branch points and cuts.

Calculation of the eq (3-7)at a large distance from the origin is made by use of the steepest descent method. The virtual friction tending to zero, the path of integration is shown in Fig. 5.

Under the above condition, the solution of the integral (3-7) is given by

$$\zeta_1=2\pi i\sum_{n=k}^{m}\frac{B(q_1^2-l_0l_n)}{g'(-l_n)}\cdot S(l_0-l_n)\cdot e^{-il_nx-\sqrt{(l_n^2-q_2^2)}\cdot(y-L)+i\sigma t}$$

$$+\frac{i-1}{\sqrt{2}}B\frac{(q_1^2-l_0q_2\sin\theta)S(l_0-q_2\sin\theta)}{(h_2/h_1)q_2\cos\theta\cos(\alpha_1L)_A+i(\alpha_1)_A\sin(\alpha_1L)_A}$$

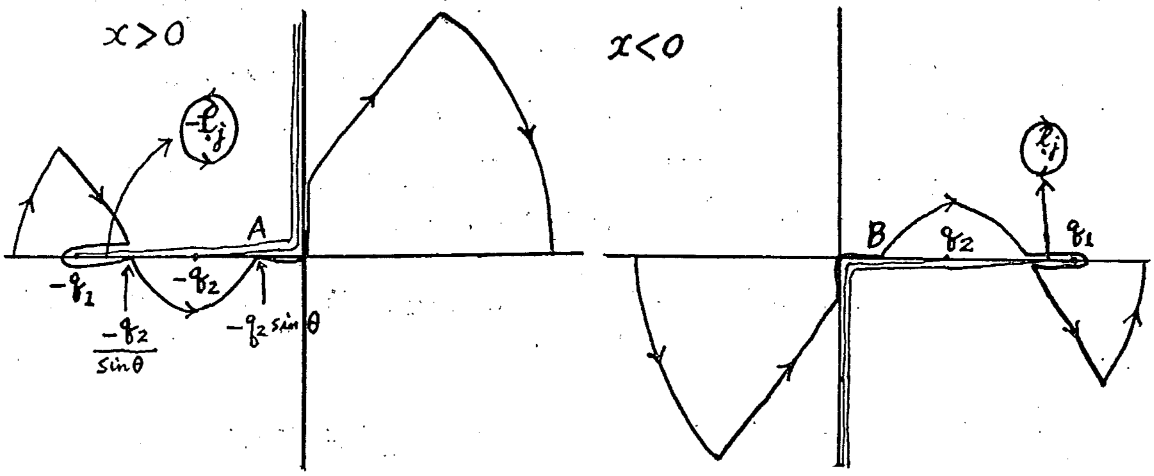

Fig. 5 The path of integration for eq. (3-7).

$$\cdot\sqrt{\left(\frac{2\pi q_2}{r}\right)}\cdot\cos\theta\cdot e^{-irq_2+i\sigma t},\ \text{for } x>0 \text{ and } \theta\geqq\sin^{-1}\left(\frac{q_2}{l_k}\right). \tag{3-9}$$

$$\zeta_1=2\pi i\sum_{n=k}^{m}\frac{B(q_1^2+l_0l_n)}{g'(-l_n)}\cdot S(l_0+l_n)\cdot e^{il_nx-\sqrt{(l_n^2-q_2^2)}\cdot(y-L)+i\sigma t}$$

$$+\frac{i-1}{\sqrt{2}}\cdot\frac{B(q_1^2-l_0q_2\sin\theta)\cdot S(l_0-q_2\sin\theta)}{(h_2/h_1)q_2\cos\theta\cos(\alpha_1L)_B+i(\alpha_1)_B\sin(\alpha_1L)_B}$$

$$\cdot\sqrt{\left(\frac{2\pi q_2}{r}\right)}\cdot\cos\theta\cdot e^{-irq_2+i\sigma t},\ \text{for } x<0 \text{ and } |\theta|\geqq\sin^{-1}\left(\frac{q_2}{l_k}\right). \tag{3-10}$$

where $x=r\sin\theta$, $y=L+r\cos\theta$; $A, B=-q_1\sin\theta$,

$$g'(-l_n)=\left[\left(\frac{h_2}{h_1}\right)\cdot\frac{1}{\sqrt{l_n^2-q_2^2}}+L\right]l_n\cos L\sqrt{q_1^2-l_n^2}$$

$$+\left[\left(\frac{h_2}{h_2}\right)L\sqrt{l_n^2-q_2^2}+1\right]\frac{l_n}{\sqrt{q_1^2-l_n^2}}\sin L\sqrt{q_1^2-l_n^2}\,.$$

and $(\ )_{A \text{ or } B}$ denotes the value when k is replaced by A or B.

The path of integration to calculate eq (3-8) at a large distance from the origin is shown in Fig. 6.

Thus, in this case the solution of eq (3-8) is given by

$$\zeta_1'=2\pi i\sum_{n=0}^{m}\frac{B\cdot(q_1^2-l_0l_n)S(l_0-l_n)}{g'(-l_n)}\cdot e^{-il_nx+i\sigma t}$$

$$\cdot\left[\cos(\alpha_1)_n(y-L)-i\left(\frac{h_2\alpha_2}{h_1\alpha_1}\right)_n\sin(\alpha_1)_n(y-L)\right]$$

$$+iB\int_{-q_2}^{+i\infty}\frac{2(q_1^2+l_0k)S(l_0+k)(h_2/h_1)\alpha_2}{\left(\frac{h_2}{h_1}\right)^2\alpha_2^2\cos^2\alpha_1L+\alpha_1^2\sin^2\alpha_1L}\cdot e^{ikx}\cdot\cos\alpha_1y\,dk,\ \text{for } x>0. \tag{3-11}$$

$$\zeta_1'=2\pi i\sum_{n=0}^{m}\frac{B(q_1^2+l_0l_n)S(l_0+l_n)}{g'(-l_n)}\cdot e^{il_nx+i\sigma t}$$

$$\cdot\left[\cos(\alpha_1)_n(y-L)-i\left(\frac{h_2\alpha_2}{h_1\alpha_1}\right)_n\sin(\alpha_1)_n(y-L)\right]$$

$$+\text{contribution by the integral along } \Gamma+q_2, \text{ for } x<0, \qquad (3\text{-}12)$$

where ($)_n$ denotes the value when k is replaced by l_n.

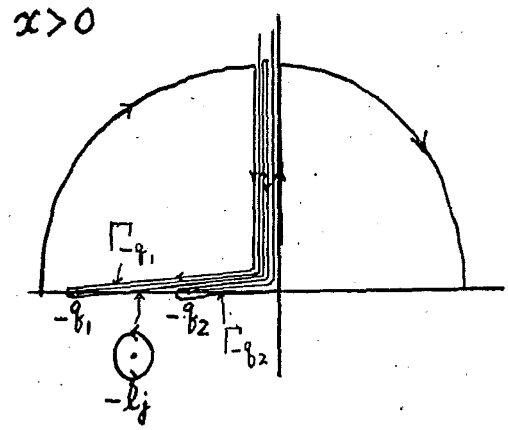

Fig. 6 The path of integration for eq. (3-8).

The last terms in the right-hand side of the eqs(3-11) and (3-12) arise from the integration along the path Γ–q_2 and Γq_2 respectively, which may tend to zero as x tends to infinity.*

3.-1.3 Powers emitted by genetaeed waves**

Power of waves is given by the product $p\,V\!\!\!/$ of the pressure p and the orbital velocity $V\!\!\!/$.

Hence, the total powers P_n^+ and P_n^- transmitted by the generated wave of the n-th mode in the positive and negative x-direction are given respectively by

$$P_n^+=\int_0^\infty \langle\overline{hpV\!\!\!/}\rangle_x dy=\frac{\pi^2\rho g^2}{\sigma}\cdot\frac{B^2l_nh_1}{\{g'(-l_n)\}^2}\cdot(q_1^2-l_0l_n)^2\,|S(l_0-l_n)|^2$$

$$\cdot\left[\frac{h_2}{h_1}\cdot\frac{1}{\sqrt{l_n^2-q_2^2}}+\left\{1-\left(\frac{h_2\alpha_2}{h_1\alpha_1}\right)_n^2\right\}L+\left\{1+\left(\frac{h_2\alpha_2}{h_1\alpha_1}\right)_n^2\right\}\frac{\sin 2(\alpha_1)_nL}{2(\alpha_1)_n}\right.$$

$$\left.+i\left(\frac{h_2\alpha_2}{h_1\alpha_1}\right)_n\cdot\frac{1-\cos 2(\alpha_1)_nL}{(\alpha_1)_n}\right], \qquad (3\text{-}13)$$

$$P_n^-=P_n^+(l_n\to -l_n)$$

Also, the total power $P_{f.w.}$ of the free radiating wave from the disturbing center is given by

$$P_{f.w.}=\int_{-\pi/2}^{\pi/2}P_r(\theta)d\theta \qquad (3\text{-}14)$$

where $P_r(\theta)$ is given by

$$P_r(\theta)=\frac{\pi\rho g^2}{\sigma}\cdot\frac{h_2B^2(q_1^2-l_0q_2\sin\theta)^2\cos^2\theta\cdot q_2^2\cdot|S(l_0-q_2\sin\theta)|^2}{\left(\frac{h_2}{h_1}\right)^2q_2^2\cos^2\theta\cos^2(\alpha_1L)_A+(\alpha_1)_A^2\sin^2(\alpha_1L)_A}. \qquad (3\text{-}15)$$

The total power transmitted from the negative x-direction by the primary edge wave P_0^- (wave number l_0) is given by

* Evaluation of these integrals will be that these to infinity as x^{-2} and in special circumstances as x^{-1} (See Ewing et al., 1957).

** Space average is made to let powers of edge waves and a free one be independent.

$$P_0^- = \frac{\rho g^2 B^2 l_0 h_1}{4\sigma} \cdot \left[\frac{h_2}{h_1} \cdot \frac{\cos^2 s'L}{s} + L + \frac{\sin 2s'L}{2s'} \right] \tag{3-16}$$

The function $|S(\mu)|^2$ in the above expressions is related to the variance spectrum $W(\mu)$ ($W(\mu)$ is defined here as $\langle\!\langle \lambda^2(x) \rangle\!\rangle = \int_{-\infty}^{+\infty} W(k)dk$) of the boundary in the following form

$$W(k) = \frac{2\pi |S(k)|^2}{L_b} \tag{3-17}$$

where L_b is the dimension of the finite patch of the irregular boundary.

3.-2 General extension of the results of sec. 3.-1

In sec. 3.-1, we have treated the simplest case that the ocean bed is of uniform depth having a continental shelf of constant depth. In general, however, the edge wave behavior will be considerably modified when we take into account of the bottom slope inside the shelf.

3.-2.1 Formulation of the problem and its solutions

For this purpose, we employ the ocean model of sec. 2.-2 (see Fig. 2). The equations for long waves are given as in sec. 2-2.
We assume here that all the relevant quantities are proportional to $e^{i(\sigma t - lx)}$, then we readily have

$$\left.\begin{aligned} u &= \frac{gl}{\sigma}\zeta, \\ v &= -\frac{g}{i\sigma} \cdot \frac{d\zeta}{dy}\zeta, \\ -i\sigma\zeta &= -ilhu + \frac{dh}{dy} \cdot v + h \cdot \frac{dv}{dy}. \end{aligned}\right\} \tag{3-18}$$

From these equations, we have the equation for ζ:

$$\frac{d^2\zeta}{dy^2} + \frac{1}{h} \cdot \frac{dh}{dy} \cdot \frac{d\zeta}{dy} + \left(\frac{\sigma^2}{gh} - l^2 \right)\zeta = 0 \tag{3-19}$$

Now, we determine the 1st order waves converted from the 0th order edge wave of incidence, when this wave travels along the coast of the irregular boundary. The boundary condition at the coast has the following expression

$$v_1(c) = u_0(c)\frac{d\lambda}{dx} - \lambda \left. \frac{\partial v_0}{\partial y} \right|_c . \tag{3-20}$$

Now we assume that the 0th order edge wave is given by

$$\zeta_m^{(0)} = \varphi_m(y) e^{i(\sigma t - l_m x)} \tag{3-21}$$

where φ_m is expressed by eq. (2-19).

Then, from the same boundary conditions as in sec. 3.-1, we have the 1-st order solution in the following form:

$$\zeta_1^{(1)} = \int_{-\infty}^{+\infty} e^{i(\sigma t - lx)} (a_l^{(1)} Y_l^{(1)}(y) + b_l^{(1)} Y_l^{(2)}(y)) dl, \tag{3-22}$$

$$\zeta_2^{(1)}=\int_{-\infty}^{+\infty} c_l^{(1)}\cdot e^{-\alpha_l y+i(\sigma t-lx)}dl,$$

where $\alpha_l=\sqrt{l^2-\frac{\sigma^2}{c_2^2}}$, $c_2^2=gh_2$ and $R_e\alpha_l\geqq 0$,

and $a_l^{(1)}$, $b_l^{(1)}$, $c_l^{(1)}$ are rendered by

$$a_l^{(1)}=\frac{1}{G(l)}\cdot[l_m(l_m-l)\varphi_m(c)-\varphi_m''(c)]S(l_m-l)[Y_l^{(2)\prime}(L)+\alpha_l Y_l^{(2)}(L)]$$

$$b_l^{(1)}=-\frac{1}{G(l)}[l_m(l_m-l)\varphi_m(c)-\varphi_m''(c)]S(l_m-l)[Y_l^{(1)\prime}(L)+\alpha_l Y_l^{(1)}(L)] \quad (3\text{-}23)$$

$$c_l^{(1)}=\frac{e^{\alpha_l L}}{G(l)}\cdot[l_m(l_m-l)\varphi_m(c)-\varphi_m''(c)]S(l_m-l)[Y_l^{(1)}(L)Y_l^{(2)\prime}(L)-Y_l^{(2)}(L)Y_l^{(1)\prime}(L)]$$

where $S(\mu)$ is the boundary spectrum as before.

Now, let the real roots of $G(l)=0$ be denoted as before by $\pm l_1$, $\pm l_2$,...... For large positive x, we have the required solution by the integral around the pole l_n:

$$\zeta_1^{(1)}=2\pi i\cdot\frac{[l_m(l_m-l_n)\varphi_m(c)-\varphi_m''(c)]S(l_m-l_n)}{G'(l_n)}\cdot[(Y_{ln}^{(2)\prime}(L)+\alpha_{ln}Y_{ln}^{(2)}(L))Y_{ln}^{(1)}(y)$$
$$-(Y_{ln}^{(1)\prime}(L)+\alpha_{ln}Y_{ln}^{(1)}(L))Y_{ln}^{(2)}(y)]e^{i(\sigma t-l_n x)} \quad (3\text{-}24)$$
$$\zeta_2^{(1)}=2\pi i\cdot\frac{[l_m(l_m-l_n)\varphi_m(c)-\varphi_m''(c)]S(l_m-l_n)}{G'(l_n)}\cdot[Y_{ln}^{(1)}(L)Y_{ln}^{(2)\prime}(L)-Y_{ln}^{(2)}(L)Y_{ln}^{(1)\prime}(L)]$$
$$\times e^{-\alpha_{ln}(y-L)+i(\sigma t-l_n x)}.$$

We readily obtain the solution for the region $x<0$ if we replace l_n by $-l_n$. The eq (3-24) represents just the edge wave of the n-th mode. Unifying the expressions (3-24) by using the function φ_n, we get

$$\zeta^{(1)}=2\pi i\cdot\frac{[l_m(l_m-l_n)\varphi_m(c)-\varphi_m''(c)]S(l_m-l_n)}{G'(l_n)}$$
$$\cdot\frac{[Y_{ln}^{(1)}(L)Y_{ln}^{(2)\prime}(L)-Y_{ln}^{(1)\prime}(L)Y_{ln}^{(2)}(L)]}{\varphi_n(L)}\cdot\varphi_n(y)e^{i(\sigma t-l_n x)} \quad (3\text{-}25)$$

Now the power of the wave averaged in time and space in the x direction is given by

$$\langle\overline{pu}\rangle=\frac{\rho g^2 l_n}{2\sigma}|\zeta|^2.$$

Hence, for the total power of the wave in the positive $x-$ direction we have

$$P_n^+=\int_{-c}^{+\infty}\langle\overline{hpu}\rangle\,dy=\frac{2\pi^2\rho g^2 l_n}{\sigma}$$
$$\cdot\frac{|l_m(l_m-l_n)\varphi_m(c)-\varphi_m''(c)|^2|S(l_m-l_n)|^2|Y_{ln}^{(1)}(L)Y_{ln}^{(2)\prime}(L)-Y_{ln}^{(1)\prime}(L)Y_{ln}^{(2)}(L)|^2}{|G'(l_n)|^2\cdot|\varphi_n(L)|^2}$$
$$\times\int_c^{\infty}h|\varphi_n(y)|^2dy$$

But now

$$\int_c^\infty h\varphi_m^* \varphi_n dy = \begin{cases} \dfrac{h(L)}{2\,l_m}\cdot c_{l\,m}^2\cdot\dfrac{G'(l_m)}{H_{lm}/\alpha_{lm}}\cdot e^{-2\alpha_{lm}L}, & (n=m)^* \\ 0. & (n\neq m) \end{cases} \quad (3\text{-}24)$$

where

$$H_{lm}=Y_{lm}^{(1)\prime}(L)Y_{lm}^{(2)\prime}(c)-Y_l^{(1)\prime}(c)Y_l^{(2)\prime}(L).$$

Hence, the edge waves are mutually orthogonal under the above defined orthogonal condition. If we apply the orthonormal function, the

$$c_l^2=\frac{2\,l_m}{h(L)}\cdot\frac{H_{lm}/\alpha_{lm}}{G'(l_m)}\cdot e^{+2\alpha_{lm}L}$$

and

$$\varphi_m(L)=\sqrt{\frac{2\,l_m}{h(L)}\cdot\frac{H_{lm}/\alpha_{lm}}{G'(l_m)}} \quad (3\text{-}25)$$

Then, we have the following expression for P_n^+

$$P_n^+=h(L)\frac{\pi^2\rho g^2}{\sigma}\cdot\frac{|\,l_m(l_m-l_n)\varphi_m(c)-\varphi_m''(c)\,|^2\cdot|\,S(l_m-l_n)\,|^2}{G'(l_n)\cdot H_{ln}/\alpha_{ln}}$$

$$\cdot\,|\,Y_{ln}^{(1)}(L)Y_{ln}^{(2)\prime}(L)-Y_{ln}^{(2)}(L)Y_{ln}^{(1)\prime}(L)\,|^2, \quad (3\text{-}26)$$

for the region $x>0$. The solution P_n^- for the negative x is obtained by replacing l_n by $-l_n$.

Also, the incident power $P_m^{(0)}$ is given by

$$P_m^{(0)}=\frac{\rho g^2 l_m}{2\sigma} \quad (3\text{-}27)$$

since φ_m is normalized to unity.

A free scattered wave at a large distance from the disturbing center is obtained by the steepest descent path method in the quite similar way as in sec. 3.-1. The result is given by

$$\zeta_{\text{f. w.}}=\frac{[l_m(l_m-q_2\sin\theta)\varphi_m(c)-\varphi_m'(c)]S(l_m-q_2\sin\theta)}{G(q_2\sin\theta)}$$

$$\cdot K(q_2\sin\theta)\cdot\frac{1+i}{\sqrt{2}}\cdot\sqrt{\frac{2\,\pi q_2}{r}}\cos\theta e^{i(\sigma t-q_1 r)} \quad (3\text{-}28)$$

where $\quad x=r\sin\theta,\quad y-L=r\cos\theta$

and $\quad K(l)=Y_l^{(1)}(L)Y_l^{(2)\prime}(L)-Y_l^{(2)}(L)Y_l^{(1)\prime}(L).$

Hence, the total power of the free scattered wave is given by

$$P_{\text{f. w.}}=\int_{-\pi/2}^{\pi/2}P_r(\theta)d\theta$$

where

* This equation is derived easily first by subtracting the differential equation for φ_m^* multiplied by φ_n from that for φ_n multiplied by φ_m^* and then by the limit process $n\to m$.

$$P_r(\theta)=\frac{\pi\rho g^2(q^2{}_2)}{\sigma}\cdot\frac{h_2\,|\,l_m(l_m-q_2\sin\theta)\varphi_m(c)-\varphi''{}_m(c)\,|^2\cdot|\,S(l_m-q_2\sin\theta)\,|^2}{|\,G(q_2\sin\theta)\,|^2}$$
$$\cdot\,|\,K(q_2\sin\theta)\,|^2\cdot\cos^2\theta \qquad (3\text{–}29)$$

In the following article, the numerical examples will be given for the power ratio $P_n^-/P_m^{(0)}$ as applications of sec. 3.-1 and 3.-2.

For the physical interperetation of the edge wave, we may conclude, because of no damping in the direction of wave progress, that it is the wave of total reflection both at the coast and at the edge of the shelf, proceeding along the coast inside the shelf in the sense of ray theory.

In the solutions, we have neglected the terms arising from the integral along the cut paths, which may be interpreted as composed of waves which reflect totally at the coast and partially at the edge of the shelf (See, Ewing et al., 1957). This will, however, be justified from the following reason: The edge waves, the free radiating wave, or the integral along the cut paths satisfies the wave equation independently of others. The former two solutions also satisfy the boundary conditions independently, but the last does not. Since there may be no other term as $x\to\pm\infty$, the last term must tend to zero. That is, the waves, which are very complicated around the disturbing area, tend to be arranged gradually as they propagate farther, and finally the solution will be composed only of the edge waves and the free radiating wave.

In the following, application will be made in case when the water depth is given by $h=by^2$ in the region 1. From eq (3-19) and the depth given here, we have the equation for $Y_l^{(k)}(y)$ in the following form:

$$\frac{d^2\zeta}{dy^2}+\frac{2}{y}\cdot\frac{d\zeta}{dy}+\left(\frac{\sigma^2}{gby^2}-l^2\right)\zeta=0.$$

On making a transformation

$$\zeta=y^{-\frac{1}{2}}w$$

then we have eq. for w such that

$$\frac{d^2w}{dy^2}+\frac{1}{y}\cdot\frac{dw}{dy}+\left(\frac{\sigma^2/gb-\frac{1}{4}}{y^2}-l^2\right)w=0.$$

Setting $z=ily$ and $\nu^2=\frac{\sigma^2}{gb}-\frac{1}{4}$, then we have the equation in the following from

$$\frac{d^2w}{dz^2}+\frac{1}{z}\cdot\frac{dw}{dz}+\left(1+\frac{\nu^2}{z^2}\right)w=0, \qquad (3.30)$$

and the solutions are

$$\begin{aligned} w&=I_{i\nu}(ly) \text{ or } I_{-i\nu}(ly) && \text{when } \nu^2>0, && (3\text{–}31)\\ &=I_{|\nu|}(ly) \text{ or } I_{-|\nu|}(ly) && \text{when } \nu^2<0 \text{ and } |\nu| \text{ is not an integer,}\\ &=I_{|\nu|}(ly) \text{ or } K_{|\nu|}(ly) && \text{when } \nu^2<0 \text{ and } |\nu| \text{ is an integer.}\end{aligned}$$

In general, the actual ocean bed seems to vary from the shore towards the sea in such a form that ν^2 should be positive so long as we assume the wave period of several ten minutes, for instance, as an example shows of the sea bed off the Sanriku Districts, where b is of the order 10^{-9} (c. g. s. unit) This case will impose a rather tedious computation owing to the lack of tables of the functions.

3.-3 Numerical examples

Here, will be given the examples of the sec. 3.-1 and 3.-2

For the variance spectrum of the coastal boundary, we assume temporarily the following form

$$W(k)=A/k^2, \tag{3-32}$$

which is the same form as Cox has obtained in the analysis of the ocean bed in the Atlantic.

(a) Example of sec. 3.-1

As constants, we adopt the following: $h_1=1000\,\mathrm{m}$, $h_2=4000\,\mathrm{m}$, $L=60\,\mathrm{km}$ and the wave period $T=30\,\mathrm{min}$. This case gives only one edge wave mode with wave number $l_1=2.74\times10^{-7}$. Then we have the power ratio of the generated edge wave to the incident one in the following form

$$P_1^-/P_0^-=-1.56\times10^{-12}\,AL_b. \tag{3-33}$$

The power ratio of the free scattered wave to the incident one is likewise given by

$$P_{\mathrm{f.\,w.}}/P_0^-=\int_{-\pi/2}^{\pi/2}R(\theta)d\theta=-5.06\times10^{-13}\,AL_b, \tag{3-34}$$

where $R(\theta)=P_r(\theta)/P_0^-$ indicates the relative power of the scattered wave in the direction θ, and has the scattering characteristics shown in Fig. 7. Here A and L_b are taken in c. g. s. unit. In this case the power of the scattered

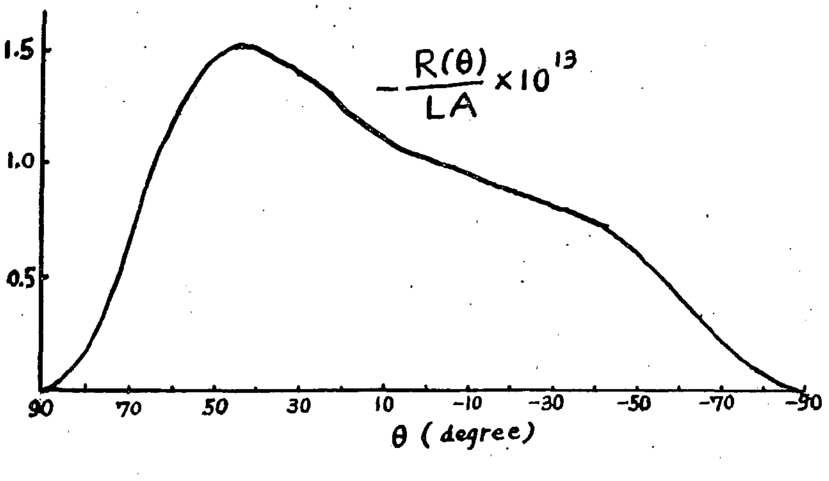

Fig. 7 The scattering characteristics.

wave is about one-third of that of converted edge wave and the damping is mainly determined by the latter effect (in this case the conversion is also regarded as the reflection). To determine the power transmitted to the positive x direction by the edge wave generated, the adopted spectral form is inadequate since it is singular at $k=0$.

To estimate the constant A, let us assume that the amplitude of the undulation of the coast is 20 km for the wave length 200km. Then we may have $A\approx 6\times 10^5$. In this case, the damping is surprisingly large, and only $L_b\approx 10$ km gives the fully scattered state, so that the perturbation procedure may fail.

(b) Example of sec. 3. 3

For simplicity, we consider the case $\nu=0$. Numerical constants are given as follows: $c=10$ km, $L=50$ km, $h_1\doteqdot 160$ m, $h_2=4$ km, $T\doteqdot$ 52min.. This case gives only one edge wave mode in the long wave range with $l_1=1.1\times 10^{-7}$, and the power ratio of edge wave generated to the incident one is given by

$$P_1^{-}/P_1^{(0)}=10^{-14}\,AL_b(10^{-14}AL_b) \tag{3-35}$$

Then, for instance, when we apply the value $A\approx 6\times 10^5$, then the edge wave suffers the damping at the rate of 6% decrease by 100 km advance in the energy flow.

In the examples given above, the larger damping in the former by the order 10^2 may be due to the deeper depth at the coast, since the coastal wall is fully responsible for the reflection of the onshore wave and hence the scattering when the coast is irregular.

4. Concluding Remarks.

So far, we have presented the general expressions and the special examples on generation and damping of edge waves. Power concept after Dr. Cox has adopted throughout the present paper to grasp the edge wave characteristics on the whole and to make the treatment easy especially in sec. 3 of the damping problem. Simple kinematics may connect it with the wave energy with relative case though it was not discussed here.

It is noted that the generation of edge waves strongly depends on the distance of the disturbance center from the shelf in the exponential form when the center is in the open sea, and the conversion damping is proportional to the variance spectrum, although, of course, the geometrical structure of the ocean bed may considerably modify the results. Certainly our adoptation of the numerical constants in the above examples will be inadequate in featuring the actual ocean bed, but this may be corrected easily at the expense of more tedious computation. The essential drawback of the present theory will be that we have completely neglected the effect of the non- linear processes such as breaking and overflowing taking place near the coast. Our hesitation in specifying depth model near the coast as well as the irregularity of coast is exclusively ascribed to this point. One may attempt to avoid this difficulty by defining that edge wave is the one which never suffers breaking near the coast, and hence has finite amplitude from Stoker's conjecture. In this case, however, still the question may remain what is the condition for the wave not to break. Investigations made by Greenspan (1958) and Carrier & Greenspan (1958) might be trials to clear out the obscure character near coast of edge waves.

Here, it must be noted that the edge waves may be interpreted as the waves constructed by interference between plane waves undergoing total reflection in the shelf (for instance, Ewing et al., 1957). Hence, their generation is essentially due to the diffraction effect of the incident wave, and in general the percentage of the generation will be strongly dependent upon the relation of curvature between the shelf edge and the incident wave front. In addition, although it may be less important, coastal irregularity will participate in the generation problem, since scattering due to the irregularity produces the wave components of the total reflection in the shelf.

At the end of this paper, we infer the edge wave phase in the record of tsunami at the coast. From the consideration of the minimum group velocity given from the dispersion curves, for instance, by Kanai (Ewing et al., 1957), it seems that the edge wave phase, at least in its first few modes, is confined in the so called Main-phase in the tsunami record. Unfortunately, the fact that the period of the edge wave is of the same order as those of shelf and bay oscillations, makes it difficult to discern the edge wave phase in the actual mareogram, and the detailed analysis is left to the future.

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