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# ALGORITHM FOR DETERMINATION OF THE EARTH ROTATION PARAMETERS AND GEODETIC COORDINATES BY USING SATELLITE LASER RANGING DATA†

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**Abstract**

An algorithm for estimation of geodetic parameters by means of a linear estimation theory applying to Satellite Laser Ranging (SLR) data was presented in the former paper of this publication series by the author (Sasaki 1984). The procedure and expression to determine the precise position of SLR site in the geocentric rectangular coordinate, the geocentric constant of gravitation ($GM$), the dynamical form factor of the earth ($J_2$) and a ballistic air-drag coefficient ($\beta$) were given. A preliminary results for the precise location of the Simosato Hydrographic Observatory and the datum shift correction from the Tokyo Datum to a global geodetic system were also presented in the paper by applying the algorithm to SLR data obtained in worldwide observation sites.

In succession from the former paper an algorithm to have results of the earth rotation parameters of pole position ($x_p, y_p$) and excessive angular velocity ($\Delta\omega$) and geodetic coordinates of observation sites expressed in latitude, longitude and height from a reference ellipsoid ($\varphi, \lambda, h$) is given in this paper in order to use the algorithm for the analysis of SLR data in the Hydrographic Department and so on.

**Key words :** Satellite laser ranging, Earth rotation, Geodetic coordinate

## 1. Introduction

An algorithm to determine a satellite orbit and geodetic parameters by means of linear estimation theory applying to Satellite Laser Ranging (SLR) data was described in a former paper of the author (Sasaki 1984). After that, SLR data have been well used for determination of the earth rotation parameters especially in the main campaign of MERIT Project from September 1983 to October 1984. The validity of the SLR together with the Very Long Baseline Interferometry (VLBI) and the Lunar Laser Ranging (LLR) was proved in the campaign (e.g. Feissel 1986). An recommendation to establish the International Earth Rotation Service (IERS) as a new organization for determination of earth rotation parameters by using these new space techniques and for its services to users instead of BIH and IPMS on January 1st of 1988 was adopted in the general meetings of both IAU in 1985 and IUGG in 1987. The Hydrographic Department of Japan (JHD) had decided to make the Simosato Hydrographic Observatory stand as a candidate for SLR observation site for IERS and this proposal was accepted by the directing board of IERS. The board also recommended that JHD accepts one of the analysis centers of SLR data in the future. To estimate the earth rotation parameters in JHD by using SLR data according to the recommendation, an algorithm to determine the earth rotation parameters given in this paper will be available.

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Owing to the improvement of ranging precision and increment of the number of SLR data obtained in the world it has been becoming possible to detect baseline length changes between SLR sites and therefore to detect the plate motions and crustal movements. In the procedure for these detections by using SLR data it has been better to fix one of components of geodetic latitude, longitude or height from a reference ellipsoid for a few SLR sites. For such a purpose an estimation algorithm to give components of the geodetic coordinate will be also available.

The expression to determine the pole position, $(x_p, y_p)$, and excessive angular velocity of the earth, $\Delta\omega$, and latitude, longitude and height from a reference ellipsoid, $(\varphi, \lambda, h)$, by using a linear estimation theory and SLR data is given in this paper.

## 2. A brief description of linear estimation procedure and notation

The equations of motion of a satellite around the earth in a non-rotating coordinate are given as

$$\dot{\boldsymbol{r}}=\boldsymbol{v},\ \dot{\boldsymbol{v}}=-\mu\frac{\boldsymbol{r}}{r^3}+\boldsymbol{R}\ (\boldsymbol{r},\ \boldsymbol{v},\ \boldsymbol{\alpha},\ \boldsymbol{\beta},\ t) \tag{1}$$

where $\boldsymbol{r}$ : position vector of a satellite, $\boldsymbol{v}$ : velocity vector, $\mu$ : geocentric constant of gravitation ($=GM$), $\boldsymbol{R}$ : perturbation acceleration, $\boldsymbol{\alpha}$ : a set of given model parameters, $\boldsymbol{\beta}$ : a set of constant unknown parameters and time, $t$. The equations (1) and $\dot{\boldsymbol{\beta}} = \boldsymbol{o}$ can be rewritten by using an n-dimensional state vector, $X$, which denotes the position and velocity of the satellite and the geodetic parameters to be estimated as following :

$$\dot{X}=F(X,t), \qquad \textit{initially}\ X(t_0)=X_0. \tag{2}$$

The state vector is related to observed values non-linearly. An $m$-dimensional $i$-th observation vector, $Y_i$, observed at $t_i$ is expressed with an error vector $\varepsilon_i$

$$Y_i=G(X_i,t_i)\ +\Delta R_i\ +\varepsilon_i, \qquad i=1,\ 2,\ \cdots\cdots,\ l \tag{3}$$

where $G(X_i,t_i)$ is $m$-vector of a non-linear function relating the state and the observation. In the case of range observation if $Y_i$ is defined as $i$-th observed raw range data, $G$ is defined as the distance from a reference point of a laser ranging system at a site to the center of mass of the satellite and $\Delta R_i$ is correction to the distance by the effects of (i) site displacement by the solid earth- and ocean tide, (ii) atmospheric refraction, (iii) center of mass correction of the satellite and (iv) range offset for laser ranging hardware obtained by a system calibration.

If the difference between $X(t)$ and a reference trajectory, $X^*(t)$, is sufficiently small in the duration $t_0 < t < t_{max}$, the equations (2) and (3) can be expanded around the reference trajectory and are rewritten neglecting the term of $(X-X^*)^2$ as:

$$\dot{x}=A(t)x, \qquad \textit{initially}\ x(t_0)=x_0,\ t_0<t<t_{max} \tag{4}$$

and

$$y_i=\tilde{H}_i x_i+\varepsilon_i, \qquad i=1,\ 2,\ \cdots,\ l \tag{5}$$

where

$$x(t)=X(t)-X^*(t), \qquad A(t)=[\partial F/\partial X]^*,$$
$$y_i=Y_i-G(X_i^*,\ t_i)-\Delta R_i, \qquad \tilde{H}_i(t_i)=[\partial G/\partial X]_i^*. \tag{6}$$

The best estimate $\hat{x}_0$ of $x$ at $t=t_0$ obtained from the observations, $y_1, y_2, \cdots, y_l$ at $t=t_1, t_2, \cdots, t_l$ is given as

$$\hat{x}_0=(H^TR^{-1}H)^{-1}H^TR^{-1}y, \tag{7}$$

where

$$y=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_l \end{bmatrix}, \quad H=\begin{bmatrix} \tilde{H}_1\Phi(t_1, t_0) \\ \tilde{H}_2\Phi(t_2, t_0) \\ \vdots \\ \tilde{H}_l\Phi(t_l, t_0) \end{bmatrix} \tag{8}$$

and $\Phi$ : transition matrix as $x_i=\Phi_i(t_i,t_o)x_o$.

### 3. Expression of $A$-matrix and $H$-matrix

As the unknowns $X$, satellite position expressed by geocentric rectangular coordinate of J2000.0, $\boldsymbol{X}$, satellite velocity, $\boldsymbol{V}$, and any other geophysical/geodetic parameters can be selected. To give their expression in this paper the author chooses the following parameters which can be understood as constants in a computating duration from $t_o$ *to* $t_{end}$ : pole position $(x_p, y_p)$, excessive angular velocity, $\Delta\omega$, latitude, longitude and height from a reference ellipsoid for unknown observation sites from the 1st to the N-th as $(\varphi_j, \lambda_j, h_j)$, $j=1,2,\cdots,N$. The expression of $X$, $\dot{X}$, $A$ and $\tilde{H}_i(t_i)$ are given by the following notation in this case :

$$X=[\boldsymbol{X},\ \boldsymbol{V},\ x_p,\ y_p,\ \Delta\omega,\ \varphi_1,\ \lambda_1,\ h_1,\ \varphi_2,\ \cdots,\ h_n]^T \tag{9}$$

$$\dot{X}=F=[\boldsymbol{V},\ \boldsymbol{\alpha},\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ \cdots,\ 0]^T \tag{10}$$

$$A=\left[\frac{\partial F}{\partial X}\right]^*=\begin{bmatrix} O_3, & I_3, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \cdots, & 0 \\ \frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{X}}, & \frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{V}}, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \cdots, & 0 \\ O_3, & O_3, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \cdots, & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ O_3, & O_3, & 0, & 0, & 0, & 0, & 0, & 0, & 0, & \cdots, & 0 \end{bmatrix} \tag{11}$$

$$\tilde{H}_i(t_i)=\left[\frac{\partial G_i}{\partial X}\right]^*=\left[\frac{\partial G_i}{\partial \boldsymbol{X}}, \frac{\partial G_i}{\partial \boldsymbol{V}}, \frac{\partial G_i}{\partial x_p}, \frac{\partial G_i}{\partial y_p}, \frac{\partial G_i}{\partial \Delta\omega}, \frac{\partial G_i}{\partial \varphi_1}, \frac{\partial G_i}{\partial \lambda_1}, \frac{\partial G_i}{\partial h_1}, \frac{\partial G_i}{\partial \varphi_2}, \cdots, \frac{\partial G_i}{\partial h_n}\right]. \tag{12}$$

In the expression above, $\boldsymbol{\alpha}$, $\frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{X}}$, $\frac{\partial \boldsymbol{\alpha}}{\partial \boldsymbol{V}}$, $\frac{\partial G_i}{\partial \boldsymbol{X}}$, $O_3$, $I_3$ are given in the former paper and $\frac{\partial G_i}{\partial \boldsymbol{V}}$ is zero.

If $i$-th observation is made at $k$-th site, the range is given by

$$\begin{aligned} R^{(i)}=R_k&=\{(U-U_k)^2+(V-V_k)^2+(W-W_k)^2\}^{\frac{1}{2}} \\ &=\{(X-X_k)^2+(Y-Y_k)^2+(Z-Z_k)^2\}^{\frac{1}{2}}. \end{aligned} \tag{13}$$

The expression of $\frac{\partial G_i}{\partial x_p}, \frac{\partial G_i}{\partial y_p}, \frac{\partial G_i}{\partial \Delta\omega}, \frac{\partial G_i}{\partial \varphi_j}, \frac{\partial G_i}{\partial \lambda_j}, \frac{\partial G_i}{\partial h_j}$ are given by using $\boldsymbol{X}$ or $(X, Y, Z)$ for satellite and $\boldsymbol{X}_j$ or $(X_j, Y_j, Z_j)$ for $j$-th site coordinate expressed in the non-rotating rectangular coordinate and also $\boldsymbol{U}$ or $(U, V, W)$ for satellite and $\boldsymbol{U}_j$, or $(U_j, V_j, W_j)$ for $j$-th site coordinate in the earth fixed rectangular coordinate as followings:

$$\frac{\partial G_i}{\partial x_p}=\frac{1}{R_j}\left\{(X_j-X)\frac{\partial X_j}{\partial x_p}+(Y_j-Y)\frac{\partial Y_j}{\partial x_p}+(Z_j-Z)\frac{\partial Z_j}{\partial x_p}\right\} \tag{14}$$

$$\frac{\partial G_i}{\partial y_p}=\frac{1}{R_j}\left\{(X_j-X)\frac{\partial X_j}{\partial y_p}+(Y_j-Y)\frac{\partial Y_j}{\partial y_p}+(Z_j-Z)\frac{\partial Z_j}{\partial x_p}\right\} \tag{15}$$

$$\frac{\partial G_i}{\partial \Delta\omega}=\frac{1}{R_j}\left\{(X_j-X)\frac{\partial X_j}{\partial \Delta\omega}+(Y_j-Y)\frac{\partial Y_j}{\partial \Delta\omega}+(Z_j-Z)\frac{\partial Z_j}{\partial \Delta\omega}\right\} \tag{16}$$

The coordinate transformation from the non-rotating rectangular coordinate of J2000.0 to the earth fixed rectangular coordinate is given by $3 \times 3$ matrixes as $\boldsymbol{U} = B \cdot S \cdot N \cdot P \cdot \boldsymbol{X}$ or $\boldsymbol{U}_j = B \cdot S \cdot N \cdot P \cdot \boldsymbol{X}_j$ and $\boldsymbol{X} = P^T \cdot N^T \cdot S^T \cdot B^T \cdot \boldsymbol{U}$ or $\boldsymbol{X}_j = P^T \cdot N^T \cdot S^T \cdot B^T \cdot \boldsymbol{U}_j$, where $B$ is a $3 \times 3$ matrix for coordinate transformation from the Pseude Earth-Fixed coordinate to the Earth-Fixed coordinate for the pole motion effect, $S$ is a $3 \times 3$ matrix for coordinate transformation from the True of Date to the Pseudo Earth-Fixed coordinate for the earth rotation effect, $N$ is a $3 \times 3$ matrix for coordinate transformatin from the Mean of Date to the True of Date for the nutation effect and $P$ is a $3 \times 3$ matrix for coordinate transformation from the J2000.0 mean equator to the Mean of Date for the precession effect as given in the former paper.

The partial derivatives in equations (15), (16) and (17) are given by differentiating the transformation matrixes as

$$\frac{\partial \boldsymbol{X}_j}{\partial x_p}=P^T \cdot N^T \cdot S^T \cdot \frac{\partial B^T}{\partial x_p} \cdot \boldsymbol{U}_j \tag{17}$$

$$\frac{\partial \boldsymbol{X}_j}{\partial y_p}=P^T \cdot N^T \cdot S^T \cdot \frac{\partial B^T}{\partial y_p} \cdot \boldsymbol{U}_j \tag{18}$$

$$\frac{\partial \boldsymbol{X}_j}{\partial \Delta\omega}=P^T \cdot N^T \cdot \frac{\partial S^T}{\partial \Delta\omega} \cdot B^T \cdot \boldsymbol{U}_j \tag{19}$$

$$\frac{\partial B^T}{\partial x_p}=\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \frac{\partial B^T}{\partial y_p}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

$$\frac{\partial S^T}{\partial \Delta\omega}=\begin{bmatrix} -t^*\sin\theta^* , & -t^*\cos\theta^* , & 0 \\ t^*\cos\theta^* & , -t^*\sin\theta^* , & 0 \\ 0 & , \quad 0 & , 0 \end{bmatrix}, \tag{20}$$

where $\omega=\omega_0+\Delta\omega$, $\theta=12^h+UT1+\alpha_m+\Delta\psi\cos\varepsilon t+\Delta\omega t^* \equiv \theta^*+\Delta\omega t^*$.

When $\Delta\omega$ is given by a unit of radian/day, $t^*$ is the ellapsed time from the start of integration in fraction of day. In this case to estimate a constant excessive angular velocity, $\Delta\omega$ , for a not-so-long duration as 3 to 5 days, *UT1* at the start time of integration should be given and *(UT1 − TAI)* also should be constant after the start time in the duration. Even if the earth rotation phase has an error to the real phase at the start time the position of ascending node of the satellite orbit will be adjusted best for the given SLR data and the constant $\Delta\omega$ will be given correctly.

As for the geodetic coordinate of an arbitrary $j$-th site, $(\varphi_j, \lambda_j, h_j)$, if one of the current $i$-th observation is made at the $k$-th site, $\frac{\partial G_i}{\partial \varphi_j}$, $\frac{\partial G_i}{\partial \lambda_j}$, $\frac{\partial G_i}{\partial h_j}$ are given by using the chain rule of partial derivative and Kronecker's delta as followings :

$$\frac{\partial G_i}{\partial \varphi_j}=\left(\frac{\partial G_i}{\partial U_j}\frac{\partial U_j}{\partial \varphi_j}+\frac{\partial G_i}{\partial V_j}\frac{\partial V_j}{\partial \varphi_j}+\frac{\partial G_i}{\partial W_j}\frac{\partial W_j}{\partial \varphi_j}\right)\delta_{jk} \tag{21}$$

$$\frac{\partial G_i}{\partial \lambda_j}=\left(\frac{\partial G_i}{\partial U_j}\frac{\partial U_j}{\partial \lambda_j}+\frac{\partial G_i}{\partial V_j}\frac{\partial V_j}{\partial \lambda_j}+\frac{\partial G_i}{\partial W_j}\frac{\partial W_j}{\partial \lambda_j}\right)\delta_{jk} \tag{22}$$

$$\frac{\partial G_i}{\partial h_j}=\left(\frac{\partial G_i}{\partial U_j}\frac{\partial U_j}{\partial h_j}+\frac{\partial G_i}{\partial V_j}\frac{\partial V_j}{\partial h_j}+\frac{\partial G_i}{\partial W_j}\frac{\partial W_j}{\partial h_j}\right)\delta_{jk}. \tag{23}$$

By using relations of $(U_j,\ V_j,\ W_j)$ and $(\varphi_j,\ \lambda_j,\ h_j)$ as

$$\begin{aligned} U_j&=(N_j+h_j)\cos\varphi_j\cos\lambda_j \\ V_j&=(N_j+h_j)\cos\varphi_j\sin\lambda_j \\ W_j&=\{N_j(1-e^2)+h_j\}\sin\varphi_j \end{aligned} \tag{24}$$

where

$$N_j=A/(1-e^2\sin^2\varphi_j)^{\frac{1}{2}}, \tag{25}$$

the expression of $\frac{\partial U_j}{\partial \varphi_j}$, $\frac{\partial V_j}{\partial \varphi_j}$, $\frac{\partial W_j}{\partial \varphi_j}$, $\frac{\partial U_j}{\partial \lambda_j}$, $\cdots$, $\frac{\partial W_j}{\partial h_j}$ are given as followings:

$$\frac{\partial U_j}{\partial \varphi_j}=-(N_j+h_j)\sin\varphi_j\cos\lambda_j+\frac{\partial N_j}{\partial \varphi_j}\cos\varphi_j\cos\lambda_j \tag{26}$$

$$\frac{\partial V_j}{\partial \varphi_j}=-(N_j+h_j)\sin\varphi_j\sin\lambda_j+\frac{\partial N_j}{\partial \varphi_j}\cos\varphi_j\sin\lambda_j \tag{27}$$

$$\frac{\partial W_j}{\partial \varphi_j}=\{N_j(1-e^2)+h_j)\cos\varphi_j+\frac{\partial N_j}{\partial \varphi_j}(1-e^2)\sin\varphi_j \tag{28}$$

$$\frac{\partial N_j}{\partial \varphi_j}=\frac{e^2\cdot N_j\sin\varphi_j\cos\varphi_j}{(1-e^2\sin^2\varphi_j)} \tag{29}$$

$$\frac{\partial U_j}{\partial \lambda_j}=-(N_j+h_j)\cos\varphi_j\sin\lambda_j \tag{30}$$

$$\frac{\partial V_j}{\partial \lambda_j}=(N_j+h_j)\cos\varphi_j\cos\lambda_j \tag{31}$$

$$\frac{\partial W_j}{\partial \lambda_j}=0 \tag{32}$$

$$\frac{\partial U_j}{\partial h_j}=\cos\varphi_j\cos\lambda_j \tag{33}$$

$$\frac{\partial V_j}{\partial h_j}=\cos\varphi_j\sin\lambda_j \tag{34}$$

$$\frac{\partial W_j}{\partial h_j}=\sin\varphi_j. \tag{35}$$

For $\frac{\partial G_i}{\partial U_j}$, $\frac{\partial G_i}{\partial V_j}$, and $\frac{\partial G_i}{\partial W_j}$, a vector expression is given in the former paper as:

$$\frac{\partial G_i}{\partial U_j} = -\frac{U^T - U_j^T}{R_j}\delta_{jk}. \tag{36}$$

The unknown parameters of $x_p$, $y_p$, $\Delta\omega$, $\varphi_j$, $\lambda_j$ and $h_j$ are to be given through the procedure given above and in the former paper.

## References

Sasaki, M.: Algorithm for Determination of Satellite Orbit and Geodetic Parameters by using Laser Ranging Data and Preliminary Results of its Application. Report of Hydrogr. Researches, No. 19, p.107-133, (1984).

Feissel, M. : Observational Results on Earth Rotation and Reference System. Reports on the MERIT-COTES Campaign on Earth Rotation and Reference System, Part III, IAU/IUGG, BIH, Paris, (1986 ed).

## 衛星レーザー測距データによる地球回転パラメーターと測地座標の決定法（要旨）

佐々木　稔

1984年３月の本研究報告において筆者は，レーザー測距データを用いて衛星軌道の初期値，観測点の地心直交座標，地球重力定数（GM），地球形状力学定数（$J_2$）及び衛星軌道抵抗係数（$\beta$）の測地パラメーターを求めるアルゴリズムを与え，また，これに各国で得られた衛星レーザー測距値を適用して，世界測地系に基づく下里水路観測所の位置及び同測地系と日本測地系との間の変換量を与えた。

本報告においては，水路部等における衛星レーザー測距データの解析に供するため，前回の報告に引き続いて，極位置（$x_p$, $y_p$）及び超過自転速度（$\Delta\omega$）の地球回転パラメーターと，緯度，経度，楕円体高（$\varphi$, $\lambda$, h）で表わした観測点の測地座標を求めるためのアルゴリズムを与える。